Lévy models in finance

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PASI - Guanajuato - June 2010

Summary

General aim: describe jummp modelling in finace through some relevant issues.

- Lecture 1: Black-Scholes model
- Lecture 2: Models with jumps
- Lecture 3: Optimal stopping for processes with jumps
- Lecture 4: Symmetry and skewness in Lévy markets

Mathematical modeling in finance

We assume we have two possibilites of investment:

- A riskless asset, named *bond*, that pays a continuously compounded interest $r \geq 0$. Its evolution is modeled by

$$\frac{dB_t}{B_t} = r dt, \qquad B_0 = 1.$$

  The solution of this differential equation is

$$B_t = e^{rt}.$$

- A risky asset, denoted by

$$S_t = S_0 e^{X_t},$$

  where $\{X_t\}$ is a stochastic process defined in a probability space $(\Omega, \mathcal{F}, P)$, satisfying $X_0 = 0$.

Options

In this model we introduce a third inversion possibility, a third asset, that we call an *option*, that is a contract that pays

$$f(S_T) \tag{1}$$

at time T to its holder.

- The asset S is the *underlying*.
- If $f(x) = (x - K)^+$ we have a *call option*,
- If $f(x) = (K - x)^+$ we have a *put option*.
- When T in (1) is fixed in the contract, the option is *european*.
- In the case that T can be chosen by the holder of the option, we call it an *american* option.

Problem: How this option can be priced, what is the fair or reasonable price of this contract at $t = 0$. We begin studying

possible models for the risk asset S, specifying the stochastic process X, called the log-price.

Brownian Motion

In 1900, Louis Bachelier introduced a model for the Brownian motion (observed in the nature by Robert Brown in 1826) in order to model the evolution of asset price fluctuations in the Paris stock exchange.

The brwonian motion or Wiener process, defined in $(\Omega, \mathcal{F}, P)$ is a stochastic process $W = (W_t)_{t \geq 0}$ such that

- $W_0 = 0$,
- has continuous trajectories,
- has independent increments: if $0 \leq t_1 \leq \cdots \leq t_n$, then

  $$W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$$

  are independent random variables.
- $W_t - W_s$ is a centered gaussian random variable with variance $t - s$, i.e.

  $$W_t - W_s \sim \mathcal{N}(0, t - s).$$

Let us remind that X is a gaussian (or normal) random variable with mean μ and variance σ^2 (we denote $X \sim \mathcal{N}(\mu, \sigma^2)$) when its probability distribution is

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u-\mu)^2}{2\sigma^2}} du$$

The density is known as the gaussian bell, given by the formula

$$\phi(x|\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Some consequences

- The random variable W_t is centered normal, and has variance t.
$$W_t \sim \mathcal{N}(0, t)$$
- The increment ΔW of the process, is $\mathcal{N}(0, \Delta t)$. Let us consider $(\Delta W)^2$. We have
$$E((\Delta W)^2) = \Delta t, \qquad Var((\Delta W)^2) = 2(\Delta t)^2$$
Then, if $\Delta t \to 0$, the variance is smaller than the expectation, this means that the variable aproximates its expectation and we denote this fact by
$$(\Delta W)^2 \sim \Delta t, \quad \text{another writing:} \quad (dW)^2 = dt$$

Black Scholes model (BS)

This is a continuous time model in $t \in [0, T]$ and has two assets:

- $B = (B_t)_{t \in [0,T]}$ that evolves deterministically, as

$$\frac{dB_t}{B_t} = r dt, \qquad B_0 = 1,$$

  where r is the interest rate. B can be thought as a bond.
- The price of a stock $S = (S_t)_{t \in [0,T]}$ has a risky evolution, modelled by a random process, according to the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW, \qquad S_0 = x,$$

  where
  - μ is the *mean return*,
  - σ si the *volatility*
  - W is a Brownian motion.

Itô's Formula

- In order to give sense to the expression "dW" we review Itô's Formula.
- Black and Scholes relies on some mathematical tools, mainly stochastic calculus and partial differential equations.
- Itô's formula is a generalization of the chain rule for usual differential calculus to differentiatite processes of the form $f(W_t)$
- It resumes the new rules governing the stochastic calculus.
- Our departure point is the equality

$$(dW)^2 = dt.$$

Let $f : \mathbf{R} \to \mathbf{R}$ be a regular function (with continuous derivatives up to order 2) Taylor expansion for f gives

$$f(x) - f(x_0) = f'(x_0)\Delta x + \frac{1}{2}f''(x_0)(\Delta x)^2 + \dots$$

Usually, the second summand is neglected with respect to the first one, but in the stochastic case, denoting $x = W_t$ and $x_0 = W_{t_0}$, we have

$$(\Delta x)^2 = (\Delta W)^2 \sim \Delta t$$

because $\Delta W \sim \sqrt{\Delta t}N(0,1)$ makes

$$E(\Delta W)^2 = \Delta t, \quad Var((\Delta W)^2) = 2(\Delta t)^2 << E(\Delta W)^2.$$

The contribution of this summand is of the same order of the first one. The other terms are effectively of higher order.

Consider now a regular function $f = f(x, t)$ of two variables. With similar arguments, it can be proved that

$$f(W_t, t) - f(W_0, 0) = \int_0^t f_x(W_s, s) dW_s + \frac{1}{2} \int_0^t f_{xx}(W_s, s) ds + \int_0^t f_t(W_s, s) ds,$$

that is Itô's formula. A short notation for this formula, is

$$df(W_t, t) = f_x(W_t, t) dW_t + \frac{1}{2} f_{xx}(W_t, t) dt + f_t(W_t, t) dt.$$

Coments

- The first integral is a *stochastic integral*

$$\int_0^t f_x(W_s, s)dW_s$$

and is defined as a limit of sums of the type

$$\sum_{i=0}^{n-1} f_x(W_{t_i})(W_{t_{i+1}} - W_{t_i})$$

- The second integral is

$$\frac{1}{2}\int_0^t f_{xx}(W_s, s)ds$$

appears due to the second term in Taylor expansion, and makes the rules of stochastic calculus different from that of usual calculus.

An example of application of Itô's formula

Consider $f(x) = x^2$. We have

$$f_t = 0, \qquad f_x = f' = 2x, \qquad f_{xx} = f'' = 2.$$

We obtain

$$\begin{aligned} f(W_t) - f(W_0) = W_t^2 &= \int_0^t (2W_s)dW_s + \frac{1}{2}\int_0^t 2ds \\ &= \int_0^t (2W_s)dW_s + t, \end{aligned}$$

that is different from the formula

$$y^2 = \int_0^y (2x)dx.$$

In this case we get an additional term.

Economic Brownian Motion

Bachelier (1900) proposes to model the evoluation of a stocks through

$$L_t = L_0 + \sigma W_t + \nu t,$$

where W_t is a Brownian Motion. As W_t is gaussian, L_t can take negative values.

In 1965 P. Samuelson proposes the model

$$G_t = G_0 \exp(\sigma W_t + \nu t),$$

for the prices of a stock. G is called Economic (or Geometric) Brownian motion.

Let us see that with this definition, the process G verifies the definition of the risky asset in Black and Scholes model. As G_t is a function of W, we can apply Itô's formula, considering

$$f(x,t) = G_0 \exp(\sigma x + \nu t).$$

We have

$$G_t = f(W_t, t),$$

the partial derivatives are

$$f_x(x,t) = \sigma f(x,t), \quad f_{xx}(x,t) = \sigma^2 f(x,t), \quad f_t(x,t) = \nu f(x,t),$$

obtaining

$$dG_t = df(W_t, t) = \sigma G_t dW_t + \frac{1}{2}\sigma^2 G_t dt + \nu G_t dt.$$

If we divide by G, we obtain

$$\frac{dG_t}{G_t} = (\nu + \frac{1}{2}\sigma^2)dt + \sigma dW_t$$
$$= \mu dt + \sigma dW_t$$

where we denote $\mu = \nu + \frac{1}{2}\sigma^2$.
In conclussion, the economic brownian motion verifies the definition of the risky asset in BS model.

As $\mu = \nu + \frac{1}{2}\sigma^2$ the formula for S is

$$S_t = S_0 \exp\left[\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right]$$

Observe that the term $\frac{1}{2}\sigma^2 t$ comes from f_{xx}, the new term in Itô's formula.
Conclussion: The geometric brownian motion is the generalization of the the continuosly compound interest formula, if we add a differential noise at every moment. Let us compare

$$dB = B(rdt), \qquad dS = S(\mu dt + \sigma \mathbf{dW}).$$

Option pricing

A portfolio in BS model is a pair of stochastic processes $\pi = (a_t, b_t)$ that represents the amount a_t of bonds and b_t of shares of the stock at time t.
The value of the portfolio π at t is

$$V_t^\pi = a_t B_t + b_t S_t.$$

In order to compute the price $V(S_0, T)$ of an european option with reward $f(S_T)$ Black and Scholes proposed to construct a portfolio that resulted equivalent to hold the option. More precisely the proposed the portfolio to be (1) *replicating* for the option and (2) *self-financing*.
In a general mathematical model of a financial market if such a portfolio exists, we say that the market is *complete*.

Let us see in detail this facts. Consider a portafolio $\pi = (a_t, b_t)$ such that:

- *Replicates* the option, this means that at the excercise time T the value of the portfolio coincides with the value of the option:
$$V_T^\pi = a_T B_T + b_T S_T = f(S_T).$$
- It is *self-financing*: the variation in the value of the portfolio is a consequence only of the variation of the prices of the assets B and S (in other terms, we do not take nor put money during the period $[0, T]$). Mathematically, this condition is formulated as
$$dV_t^\pi = a_t dB_t + b_t dS_t.$$

The price at time $t = 0$ of such a portfolio is deined as the *rational* price of the option, that is

$$V(S_0, T) = a_0 B_0 + b_0 S_0.$$

Construction of the portfolio

Black and Scholes proved that the replicating and self-financing portfolio exists, and is unique, giving then the rational price of the call option.

In order to find this portfolio, we look for a function $H(x,t)$ such that

$$V_t^\pi = H(S_t, t)$$

The replicating condition is $V_T^\pi = f(S_T)$, and this condition is satisfied if

$$H(x,T) = f(x).$$

As the portfolio and the option are equivalent, the price of the option is the initial value of the portfolio, i.e. $H(S_0, 0)$. In order to determine H and $\pi = (a_t, b_t)$ such that

$$V_t^\pi = a_t B_t + b_t S_t = H(S_t, t)$$

we compute the stochastic differential of V^π by two different ways, and equate the results.

On one side, as S is a function of W, and H is a function of S, we apply Itô's formula, to obtain that

$$dV^\pi = dH = (\mu S H_x + \frac{1}{2}\sigma^2 S^2 H_{xx} + H_t)dt + H_x S\sigma dW. \qquad (2)$$

On the other side, as π is self financing, taking into account that $a_t B_t = H_t - B_t S_t$, we have

$$\begin{aligned} dV^\pi &= adB + bdS = raBdt + b(\mu Sdt + \sigma SdW) \\ &= r(H - bS)dt + \mu bSdt + bS\sigma dW. \\ &= (rH + (\mu - r)bS)\,dt + bS\sigma dW. \qquad (3) \end{aligned}$$

We now equate the coefficients of dW in (2) y (3). We obtain:

$$b_t = H_x(S_t, t).$$

Black-Scholes equation

After this we equate the coefficient in dt, and after some simple transformations, we get

$$rSH_x + \frac{1}{2}\sigma^2 S^2 H_{xx} + H_t = rH.$$

Furthermore, as we seek for a replicating portfolio, we have the additional condition $H(S_T, T) = f(S_T)$. Both conditions are verified if we find a function H such that

$$\begin{cases} \frac{1}{2}\sigma^2 x^2 H_{xx}(x,t) + rxH_x(x,t) + H_t(x,t) & = & rH(x,t) \\ H(x,T) & = & f(x) \end{cases}$$

This is Black-Scholes equation. It is partial differential equation (PDE), where the replication condition gives the border condition. The first obtained contidion

$$b_t = H_x(S_t, t)$$

is relevant also, as it gives the amount of stock necessary to replicate the option, i.e. the hedge.

It is not difficult to verify that this PDE has a closed solution, given by

$$H(x,t) = x\Phi(x_+(x,t)) - e^{-rT} K\Phi(x_-(x,t))$$

where

$$x_+(x,t) = \left(\log \frac{xe^{r(T-t)}}{K} - \frac{1}{2}\sigma^2(T-t)\right) / (\sigma\sqrt{T-t})$$

$$x_-(x,t) = \left(\log \frac{xe^{r(T-t)}}{K} + \frac{1}{2}\sigma^2(T-t)\right) / (\sigma\sqrt{T-t}).$$

Finally, the value of the option is obtained with $t = 0$, getting

$$V(S_0, T) = S_0\Phi(x_+) - e^{-rT} K\Phi(x_-)$$

with

$$x_\pm = \left(\log \frac{S_0 e^{rT}}{K} \pm \frac{1}{2}\sigma^2 T\right) / (\sigma\sqrt{T}).$$

Relevance of Black-Scholes formula

A key consequence of Black Scholes formula is that the price of the option does not depend of the mean return μ of the risky asset. There are three parameters that depend on the contract (S_0, K, T) and two parameters from the economical model: r and σ. In order to apply the formula this parameters must be determined:

- r can be obtained as the interest rate of US bonds with similar expiration time T.
- σ is not observable, in practice a value of σ is obtained form other option values quoted in the market. This is called the *implied volatility*.

Theoretical consequences of BS formula

Key observation: As we have seen, in BS formula μ does not appear, only r. Let us transform the equation for the risky asset in the following way:

$$\begin{aligned}\frac{dS}{S} &= \mu dt + \sigma dW = rdt + \sigma d\left(W_t + \frac{\mu - r}{\sigma} t\right) \\ &= rdt + \sigma dW_t^*\end{aligned}$$

were we deonte

$$W_t^* = W_t + \frac{\mu - r}{\sigma} t.$$

Here we require the help of *Girsanov's Theorem*

Risk neutral probability and Girsanov Theorem

Theorem Given a Wiener process W defined in a probability space $(\Omega, \mathcal{F}, P)$, there exists a probability measure Q such that the process

$$W_t^* = W_t + \frac{\mu - r}{\sigma} t = W_t + qt,$$

is a Wiener process under Q. Furthermore the measures P and Q are equivalent, with Radon-Nykodym density given by

$$\frac{dQ}{dP} = \exp\left(-qT - \frac{1}{2} q^2 W_T\right)$$

This suggest to consider the model

$$\frac{dB}{B} = rdt, \qquad \frac{dS}{S} = rdt + \sigma dW^*$$

in the probability space $(\Omega, \mathcal{F}, Q)$, where W^* is a Wiener process.

It is important to note that under Q the mean return of both the non-risky and risky asset is the same, r.
We have seen that the respective solutions of this equations are

$$B_t = e^{rt}, \qquad S_t = S_0 \exp\left(\sigma W_t^* + (r - \sigma^2/2)t\right)$$

Whe then have

$$\frac{S_t}{B_t} = S_0 \exp\left(\sigma W_t^* - \sigma^2 t/2\right) \text{ is a } Q\text{-martingale} \tag{4}$$

Observe that Q is the only measure that assures this property (4).
Summarizing:

- we change P by Q, μ by r, W by W^*.
- both assets B and S in the model have the same mean return r under Q,

Let us interpret the meaning of the measure Q.
In order to do this we use the following properties of the stochastic integral

- (1) $\left(\int_0^t b_t dW_t^*\right)_{t\geq 0}$ is a Q-martingale
- (2) If
$$dX_t = a_t dt + b_t dW_t^*$$
then
$$X \text{ is a } Q\text{-martingale if and only if } a_t = 0.$$

Excercise: Verify that the value of the discounted portfolio is a martingale under Q, that is

$$\frac{H(S_t, t)}{B_t} \quad \text{is a } Q\text{-martingale.}$$

Solution: We have $H/B = e^{-rT}H$. By Itô's formula,

$$d(e^{-rT}H) = e^{-rT}\left(-rHdt + dH\right)$$

As $dH = rHdt + bS\sigma dW^*$, we substitute to obtain

$$d\left(\frac{H(S_t, t)}{B_t}\right) = bS_t\sigma dW_t^*$$

verifying that the trend is null, and by property (2) we obtain that the quotient is a Q-martingale.
As the martingales preserve the expectation, we deduce that the price of the option with payoff $f(S_T)$ satisfies

$$V(x, T) = H(S_0, 0) = E_Q(e^{-rT}H(S_T, T)) = e^{-rT}E_Q(f(S_T)),$$

were we use the final condition $H(x, T) = f(x)$.
Conclussion: The price of the option in BS model is the expectation of the payoff of the option under the probability measure Q, that we call *risk-neutral probability*.

Computing BS formula for a call option

Let us compute the price $V(S_0, T)$ of a call option. We know that

$$V(x,T) = e^{-rT} E_Q(f(S_T)).$$

Under Q, with $S_0 = x$, we have

$$S_T = S_0 \exp(\sigma W_T^* - \frac{1}{2}\sigma^2 T + rT).$$

We use that

- $W_T^* \sim \sqrt{T} Z \sim \mathcal{N}(0,T)$, si $Z \sim \mathcal{N}(0,1)$.
- $\alpha = \frac{\log\left(S_0 e^{rT}/K\right) - \sigma^2 T/2}{\sigma\sqrt{T}} = x_-$.

We have

$$\begin{aligned} V(x,T) &= e^{-rT} \int_{-\infty}^{+\infty} \left(S_0 e^{\sigma\sqrt{T}u - \frac{1}{2}\sigma^2 T + rT} - K \right)^+ \phi(u) du \\ &= e^{-rT} \int_{-\alpha}^{+\infty} \left(S_0 e^{\sigma\sqrt{T}u - \frac{1}{2}\sigma^2 T + rT} - K \right) \phi(u) du \end{aligned}$$

$$\begin{aligned}
=&S_0 \int_{-\alpha}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\sigma\sqrt{T}u-\frac{1}{2}\sigma^2 T-u^2/2} du - Ke^{-rT} \int_{\alpha}^{+\infty} \phi(u) du \\
=&S_0 \int_{-\alpha}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-(u-\sigma\sqrt{T})^2/2} du - Ke^{-rT} \int_{-\infty}^{\alpha} \phi(u) du \\
=&S_0 P(Z+\sigma\sqrt{T} \geq -\alpha) - Ke^{-rT} P(\sqrt{T}Z \geq -\alpha) \\
=&S_0 P(Z \leq \alpha+\sigma\sqrt{T}) - Ke^{-rT} P(Z \leq \alpha),
\end{aligned}$$

that is the Black-Scholes formula because $\alpha+\sigma\sqrt{T}=x_+$.